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## Forced Transverse Vibration of a Solid Viscoelastic Cylinder Bonded to a Thin Casing<sup>1</sup>

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*The amplitude versus frequency-response spectra of stress and displacement components within a solid viscoelastic cylinder bonded to a thin elastic casing are obtained when arbitrary normal and tangential stresses are applied to the outer surface of the casing. Special consideration is given to assemblies whose cores are made either of a Voigt material or a Maxwell material. A quantitative comparison of the bond stress amplitude spectrum at the lowest circumferential wave number reveals for both Voigt and Maxwell cores, and for small values of retardation time and relaxation time, respectively, that the amplitude ratio between the radial bond stress and the lateral pressure decreases with increasing values of the time constant. As the time constants get larger, the resonant amplitude increases for the Maxwell material, and decreases for the Voigt material. The Voigt core essentially behaves like an elastic solid at small values of its retardation time, and like a viscous fluid at large values of its retardation time. The Maxwell core essentially behaves like a nonviscous fluid at small values of its relaxation time and like an elastic solid at large values of its relaxation time.*

### Introduction

LET US investigate the response to a forced vibration of a long solid viscoelastic cylinder bonded to a thin casing. The vibration inducing excitation consists of a time-varying surface pressure and shear stress applied to the outside of the casing in such a way that their spatial variations are only in the circumferential direction. The assembly is assumed to be long enough to be analyzed from the point of view of plane strain, so that the displacements in the longitudinal direction may be neglected.

The problem under consideration has a considerable number of technologically important applications in the field of solid propellant rocket motors. During high velocity motion of a rocket through a compressible fluid medium, surface pressures may be met which are of the pulse type, or which may be represented by stochastic functions, or which are of the periodic type. The pulse-type pressure arises from uniform waves propagated through the medium, stochastic pressures are developed from

acoustic noise, and periodic pressures arise from sources of steady vibration in the medium. The vibrations developed during each of the foregoing loadings induce stresses whose magnitudes affect the structural integrity of the assembly and represent one of the factors which has to be taken into consideration in an evaluation of the reliability of the assembly during operation.

In the present paper, equations will be derived from which one can evaluate for specific time-dependent loading programs, both the normal and shear stress at the interface between a viscoelastic cylinder and its casing, as well as equations for the radial and tangential stress and displacement components throughout the rest of the assembly. The newly derived relations will be used to obtain some numerical results for a particular set of geometric and material parameters so that a quantitative comparison can be made between a Voigt type and a Maxwell type of viscoelastic core with several different values of retardation time or relaxation time, respectively.

The forced-vibration analysis to be presented is an extension of a previous paper by the authors [1],<sup>2</sup> which dealt with the free transverse vibrations of elastic assemblies geometrically identical to the one herein considered. The extension to forced vibrations of materials with viscoelastic stress-strain properties is accomplished by the use of "a modification" to a dynamic correspondence principle suggested by Bland [2].

Some theoretical work on the forced transverse vibration of a solid, elastic cylinder has been previously presented by Baltrukonis [3]. He found the steady-state displacement field in a cylinder due to a rigid-body translation of the surface which varies sinusoidally with time. Baltrukonis' theoretical solution

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<sup>2</sup> Numbers in brackets designate References at end of paper.

was subsequently studied by Magrab [4] who plotted the distorted position of circles and rays initially inscribed on the cylinder.

Forced vibration of a hollow viscoelastic cylinder assembly has also been analyzed by Henry and Freudenthal [5]. They treated only axisymmetric motion in meridional planes and used membrane equations for the shell in considering the interaction between the inner cylinder and the outer shell.

Rogers and Lee [6] studied the effect of time-dependent loads on a compressible viscoelastic cylinder bonded to an elastic shell, by neglecting vibratory inertia forces. Achenbach [7], on the other hand considered the dynamic response of a case-bonded viscoelastic cylinder with an incompressible core.

## Differential Equations and Boundary Conditions

A typical cross section taken through the assembly is shown in Fig. 1. The viscoelastic solid propellant is designated "core" and lies in the region  $0 \leq r \leq a$ . At  $r = a$ , the core is bonded to a thin casing of wall thickness  $h$ . During forced vibration, surface loads  $p(\theta, t)$  and  $q(\theta, t)$  are applied to the outer surface of the casing in the normal and tangential directions, respectively. The radial and circumferential coordinates are designated  $r, \theta$  while the parameter time is designated  $t$ . Under load, particles are displaced in the radial direction  $u(r, \theta, t)$  and in the tangential direction  $v(r, \theta, t)$ . There is no displacement in the axial direction. The associated radial, circumferential, and shear stresses  $\sigma_{rr}$ ,  $\sigma_{\theta\theta}$ , and  $\sigma_{r\theta}$ , respectively, are shown on a typical element described in polar coordinates. With no additional applied body force on the propellant, the differential equations for the resultant displacement vector  $\bar{w}$  of the core are the same as they are for the free-vibration problem. Hence the current presentation can be shortened by referencing the fact that the authors have shown [1] the differential equation for the displacement vector  $\bar{w}$  to be

$$\left(K + \frac{4}{3}G\right) \text{grad div } \bar{w} - G \text{curl curl } \bar{w} = \gamma \frac{\partial^2 \bar{w}}{\partial t^2} \quad (1)$$

for an assembly with a compressible elastic core,<sup>3</sup> where  $K$  is the bulk modulus,  $G$  is the shear modulus, and  $\gamma$  is the mass density per unit volume of the core material;  $\sigma$  is the mean normal stress.

Thin shell theory is used to describe the deformation of the casing. The deflection of the casing middle surface, is approximately equal to the deflection of the core at  $r = a$ , because of continuity considerations. Hence the differential equations of the casing may serve as the boundary conditions for the displacement equations of the core. The required boundary conditions were derived by the authors [1] for an elastic case and core, with the case free of external surface tractions. When they are modified to include an externally applied radial pressure  $p(\theta, t)$  and a circumferential shear stress  $q(\theta, t)$  at the outer surface of the casing, the boundary conditions at  $r = a$  become

$$\begin{aligned} \frac{Eh}{a^2(1-\nu^2)} \left(u + \frac{\partial v}{\partial \theta}\right) - \frac{D}{a^4} \frac{\partial^3}{\partial \theta^3} \left(v - \frac{\partial u}{\partial \theta}\right) + \rho \frac{\partial^2 u}{\partial t^2} \\ = -p - \left(K + \frac{4}{3}G\right) \frac{\partial u}{\partial r} - \frac{1}{a} \left(K - \frac{2}{3}G\right) \left(u + \frac{\partial v}{\partial \theta}\right) \end{aligned} \quad (2a)$$

$$\begin{aligned} \frac{Eh}{a^2(1-\nu^2)} \frac{\partial}{\partial \theta} \left(u + \frac{\partial v}{\partial \theta}\right) + \frac{D}{a^4} \frac{\partial^2}{\partial \theta^2} \left(v - \frac{\partial u}{\partial \theta}\right) - \rho \frac{\partial^2 v}{\partial t^2} \\ = -q + G \frac{\partial v}{\partial r} + \frac{G}{a} \left(\frac{\partial u}{\partial \theta} - v\right) \end{aligned} \quad (2b)$$

The symbol  $E, \nu$  are, respectively, Young's modulus and Poisson's ratio for the casing material.  $D = Eh^3/12(1-\nu^2)$  is the flexural rigidity of the casing, and  $\rho$  is the mass density of the casing, per unit area of the middle surface.

<sup>3</sup> The current paper is concerned with an assembly having a compressible elastic core. The corresponding solution for an assembly with an incompressible elastic core is also presented in reference [8].

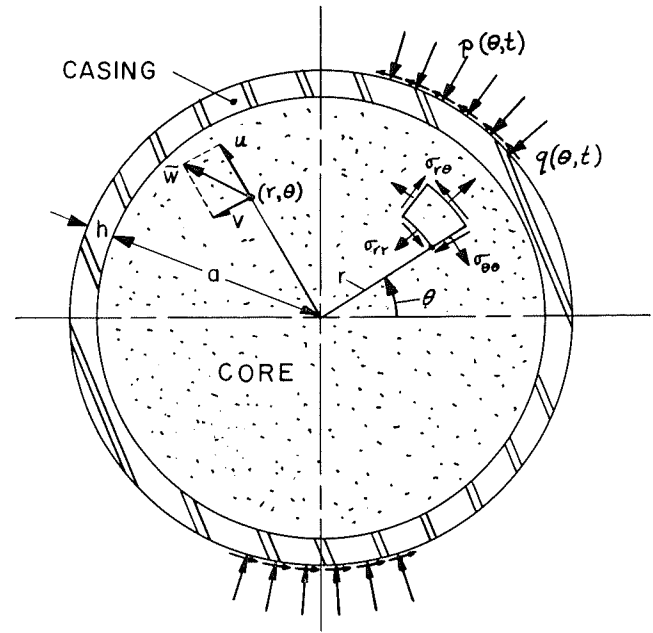


Fig. 1 A solid cylinder bonded to a thin casing

The foregoing describe the elastic field equations and appropriate boundary conditions to be used in the investigation of the forced-vibration problem.

## Bivariate Fourier Transform

To solve the system of partial differential equations presented in the previous section, use will be made of the bivariate Fourier transform, to be defined. The resulting equations will be solved and the inverse transform will then be applied. This method of solution imposes one apparently undesirable restriction; namely, the inability to impose initial conditions on the displacement vector  $\bar{w}(r, \theta, 0)$  and on the velocity vector  $\partial \bar{w}(r, \theta, 0)/\partial t$ . As a consequence of this restriction only a particular solution is obtained which nevertheless is important because of its use in computing the steady-state response to periodic inputs.

The bivariate Fourier transform will be defined and some of its properties observed by discussing an arbitrary scalar function  $f(r, \theta, t)$ . Later the transform will be applied to the displacement vector and to the stress components. The bivariate Fourier transform (or spectrum)  $F_n$  is obtained by the double integration of  $f$  with respect to  $\theta$  and  $t$ . The function  $f$  can be recovered from  $F_n$  by taking the inverse Fourier transform of  $F_n$ . Specifically, the functions  $F_n$  and  $f$  are related by equations (3a) and (3b)

$$F_n(j\omega, r) = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} e^{-j(\omega t + n\theta)} f(r, \theta, t) d\theta dt \quad (3a)$$

$$f(r, \theta, t) = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j(\omega t + n\theta)} F_n(j\omega, r) d\omega \quad (3b)$$

Many useful relations can be derived from equation (3), including the following

$$f(r, \theta, t) = f(r, \theta + 2\pi, t) \quad (4a)$$

$$\mathcal{F} \left( \frac{\partial f}{\partial t} \right) = j\omega \mathcal{F}(f); \quad \mathcal{F} \left( \frac{\partial f}{\partial \theta} \right) = jn \mathcal{F}(f) \quad (4b, c)$$

where the symbol  $\mathcal{F}(\ )$  denotes the Fourier transform of the quantity in parentheses.

It is convenient in the present analysis to introduce the Fourier transform of the vector displacement function  $\bar{w}(r, \theta, t)$ , which is equal to  $u\hat{r} + v\hat{\theta}$ , where the radial and circumferential displacements  $u$  and  $v$  are scalars;  $\hat{r}, \hat{\theta}$  are the base vectors in the radial and circumferential directions, respectively. The operation of

forming the transform of  $\bar{\mathbf{w}}$ , which is a vector, can be accomplished by first applying the  $\mathcal{F}$  operator to the scalar displacement components  $u$  and  $v$ . Should we define  $U_n$  as  $\mathcal{F}(u)$  and  $V_n$  as  $\mathcal{F}(v)$ , the vector field  $\bar{\mathbf{W}}_n$  equal to  $U_n\hat{r} + V_n\hat{\theta}$  may then be interpreted as the Fourier transform of  $\bar{\mathbf{w}}$ . The vector space associated with such an interpretation is Hermitian over the field of complex numbers.

The foregoing interpretation of the Fourier transform of the displacement vector can be applied to equation (1) in order to obtain the transformed displacement equation for an assembly with a compressible core,

$$(K + \frac{4}{3}G) \text{grad div} (\bar{\mathbf{W}}_n e^{jn\theta}) - G \text{curl curl} (\bar{\mathbf{W}}_n e^{jn\theta}) + \gamma\omega^2 \bar{\mathbf{W}}_n e^{jn\theta} = 0 \quad (5)$$

The Fourier transform may also be applied to the associated boundary conditions. In addition to the displacement components  $u$  and  $v$ , the boundary conditions contain the externally applied radial pressure  $p$ , the externally applied shear stress  $q$ , and the mean normal stress  $\sigma$ . These are all scalars too, so that the Fourier transform may be applied directly.  $\mathcal{F}(p)$  is to be called  $P_n$ ,  $\mathcal{F}(q)$  is to be called  $Q_n$ , and  $\mathcal{F}(\sigma)$  is to be called  $\Sigma_n$ ;  $U_n$  and  $V_n$  were previously defined. When the Fourier transform of equations (2) are taken, it is found that the transform of the boundary conditions at the interface  $r = a$  may be written

$$\frac{Eh}{a^2(1-\nu^2)} (U_n + jnV_n) + \frac{jn^3D}{a^4} (V_n - jnU_n) - \rho\omega^2 U_n = -P_n - \left(K + \frac{4}{3}G\right) \frac{\partial U_n}{\partial r} - \frac{\left(K - \frac{2}{3}G\right)}{a} (U_n + jnV_n) \quad (6a)$$

$$\frac{Eh}{a^2(1-\nu^2)} jn (U_n + jnV_n) - \frac{n^2D}{a^4} (V_n - jnU_n) + \rho\omega^2 V_n = -Q_n + G \frac{\partial V_n}{\partial r} + \frac{G}{a} (jnU_n - V_n) \quad (6b)$$

The general solution to equation (5) and with  $\bar{\mathbf{W}}_n$  expressed as  $U_n\hat{r} + V_n\hat{\theta}$  with  $n = 0, 1, 2, \dots$ , was previously found by the authors [1] to be

$$U_n = -\frac{jn}{r} b_n J_n(\beta r) + c_n \left[ \alpha J_{n-1}(\alpha r) - \frac{n}{r} J_n(\alpha r) \right] \quad (7a)$$

$$V_n = b_n \left[ \beta J_{n-1}(\beta r) - \frac{n}{r} J_n(\beta r) \right] + \frac{jn}{r} c_n J_n(\alpha r) \quad \text{for } n = 0, 1, 2, \dots \quad (7b)$$

where  $J_n(x)$  is Bessel's function of the first kind of order  $n$ , and

$$\alpha = \omega\sqrt{\gamma/(K + \frac{4}{3}G)}; \quad \beta = \omega\sqrt{\gamma/G} \quad (8)$$

The coefficients  $b_n$  and  $c_n$  are arbitrary constants.

Values of the foregoing Fourier transforms for negative integers  $n$  can be obtained by applying the property that the spectrum at negative  $n$  and positive  $\omega$  must be the complex conjugate of the spectrum at positive  $n$  and negative  $\omega$ . In general the spectra must be evaluated for all real frequencies  $\omega$ , from  $-\infty$  to  $+\infty$ .

Each of the general solutions contain arbitrary constants  $b_n$  and  $c_n$ . They may be expressed in terms of the input spectra  $P_n(j\omega)$ ,  $Q_n(j\omega)$  by solving the two linear, nonhomogeneous equations obtained when the general solution is substituted into the boundary conditions.

The aforementioned linear, nonhomogeneous equations may be obtained by substituting the general solution of equation (7) into the boundary conditions of equation (6). They may be written

$$M \begin{bmatrix} b_n \\ c_n \end{bmatrix} = \frac{a^5}{D} \begin{bmatrix} P_n \\ Q_n \end{bmatrix} \quad (9)$$

The term  $M$  in the previous expression is the  $2 \times 2$  matrix defined

$$M = A \begin{bmatrix} J_n(\zeta) & 0 \\ 0 & J_n(z) \end{bmatrix} + B \begin{bmatrix} \zeta J_{n-1}(\zeta) & 0 \\ 0 & z J_{n-1}(z) \end{bmatrix} \quad (10)$$

It contains the second-order matrix  $A$  whose elements  $a_{rs}$  are defined

$$a_{11} = jn \left\{ (n+1) \left[ n^3 + 12 \left( \frac{a}{h} \right)^2 \right] - \lambda\omega^2 - 2(n+1) \frac{Ga^3}{D} \right\} \quad (11a)$$

$$a_{12} = n \left\{ (n+1) \left[ n^3 + 12 \left( \frac{a}{h} \right)^2 \right] - \frac{2Ga^3}{D} - \lambda\omega^2 \right\} + \left( K + \frac{4}{3}G \right) \frac{z^2 a^3}{D} \quad (11b)$$

$$a_{21} = n \left\{ \lambda\omega^2 - n(n+1) \left[ n + 12 \left( \frac{a}{h} \right)^2 \right] \right\} - [\zeta^2 - 2n(n+1)] \frac{Ga^3}{D} \quad (11c)$$

$$a_{22} = jn \left\{ n(n+1) \left[ n + 12 \left( \frac{a}{h} \right)^2 \right] - \lambda\omega^2 - 2(n+1) \frac{Ga^3}{D} \right\} \quad (11d)$$

The preceding contain newly defined parameters

$$z = \alpha a, \quad \zeta = \beta a, \quad \lambda = \rho a^4/D$$

The matrix  $M$  also contains the second-order matrix  $B$  whose elements  $b_{rs}$  are defined

$$b_{11} = jn \left[ \frac{2Ga^3}{D} - n^2 - 12 \left( \frac{a}{h} \right)^2 \right] \quad (12a)$$

$$b_{12} = -n^4 - 12 \left( \frac{a}{h} \right)^2 + \frac{2Ga^3}{D} + \lambda\omega^2 \quad (12b)$$

$$b_{21} = n^2 \left[ 1 + 12 \left( \frac{a}{h} \right)^2 \right] - \frac{2Ga^3}{D} - \lambda\omega^2 \quad (12c)$$

$$b_{22} = jn \left\{ \frac{2Ga^3}{D} - \left[ n^2 + 12 \left( \frac{a}{h} \right)^2 \right] \right\} \quad (12d)$$

The spectra  $U_n$ ,  $V_n$  of the displacement components are evaluated by solving equation (9) for  $b_n$ ,  $c_n$  and substituting the results into equation (7). Determination of the core stress components will be discussed in the next section.

## Fourier Spectra of Core Stress Components

The stress and strain components within an elastic material obey Hooke's law. By making use of the strain-displacement relations, each of the stress components can be expressed in terms of the displacement components. For a problem of plane strain the stress-displacement relations in cylindrical coordinates may be written

$$\sigma_{r\theta} = G \left( \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right) \quad (13a)$$

$$\sigma_{rr} = \left( K + \frac{4}{3}G \right) \frac{\partial u}{\partial r} + \left( K - \frac{2}{3}G \right) \left( \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) \quad (13b)$$

$$\sigma_{\theta\theta} = \left( K - \frac{2}{3}G \right) \frac{\partial u}{\partial r} + \left( K + \frac{4}{3}G \right) \left( \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) \quad (13c)$$

$$\sigma_{zz} = \left( K - \frac{2}{3}G \right) \left( \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) \quad (13d)$$

$$\sigma_{\theta z} = \sigma_{zr} = 0 \quad (13e)$$

where the shear modulus  $G$  and the bulk modulus  $K$  are related to the moduli of elasticity  $E$  and Poisson's ratio  $\nu$  by the relations

$$G = \frac{E}{2(1 + \nu)}, \quad K = \frac{E}{3(1 - 2\nu)} \quad (14)$$

Each of the stress components are scalar functions so that it is permissible to form the bivariate Fourier transform in each case. The resulting spectra of the stress components  $\Sigma_n^{(ij)}$  is equal to  $\mathcal{F}(\sigma_{ij})$  with  $i, j$  equal to  $r, \theta, z$ . When the Fourier transform of equation (13) is taken, it is found, in view of equation (4c) that the spectra of the stress components can be expressed in terms of the spectra of the displacement components by the relations

$$\Sigma_n^{(r\theta)} = G \left( \frac{\partial V_n}{\partial r} - \frac{V_n}{r} + \frac{jn}{r} U_n \right) \quad (15a)$$

$$\Sigma_n^{(rr)} = \left( K + \frac{4}{3} G \right) \frac{\partial U_n}{\partial r} + \left( K - \frac{2}{3} G \right) \left( \frac{1}{r} \right) (U_n + jn V_n) \quad (15b)$$

$$\Sigma_n^{(\theta\theta)} = \left( K - \frac{2}{3} G \right) \frac{\partial U_n}{\partial r} + \left( K + \frac{4}{3} G \right) \left( \frac{1}{r} \right) (U_n + jn V_n) \quad (15c)$$

$$\Sigma_n^{(zz)} = \left( K - \frac{2}{3} G \right) \left[ \frac{\partial U_n}{\partial r} + \frac{1}{r} (U_n + jn V_n) \right] \quad (15d)$$

$$\Sigma_n^{(\theta z)} = \Sigma_n^{(zr)} = 0 \quad (15e)$$

Solutions for  $U_n$ ,  $V_n$ , and  $\Sigma_n$  were presented in the previous section. Hence the spectra of stress components can be determined directly from equations (15).

## Extension of Solution to Linear Viscoelastic Materials

The foregoing solution to the forced-vibration problem of an elastic assembly is easily extended to linear viscoelastic materials, by making use of "a modified form" of a dynamic correspondence principle originally derived by Bland [2]. A modification to the original work is required because of the use of the bilateral Fourier transform in the present analysis, which is not identical to that used by Bland.

The dynamic correspondence principle to be employed states that the spectrum of the viscoelastic solution can be obtained from the spectrum of the corresponding elastic solution simply by replacing the elastic constants by their corresponding viscoelastic complex moduli. To verify the dynamic correspondence principle for the present analysis, let us start with the stress-strain relations for a homogeneous isotropic linear viscoelastic core material [2].

$$P_s S_{ij} = Q_s e_{ij} \quad i, j = r, \theta, z \quad (16a)$$

$$P_\nu' \sigma_{kk} = Q_\nu' \epsilon_{kk} \quad (16b)$$

Where  $P_s$ ,  $Q_s$ ,  $P_\nu'$ ,  $Q_\nu'$  are the linear differential operators

$$P_s = \sum_{r=0}^{r=n} p_r \frac{\partial^r}{\partial t^r}, \quad Q_s = \sum_{r=0}^{r=n} q_r \frac{\partial^r}{\partial t^r} \quad (17a, b)$$

$$P_\nu' = \sum_{r=0}^{r=n} p_r' \frac{\partial^r}{\partial t^r}, \quad Q_\nu' = \sum_{r=0}^{r=n} q_r' \frac{\partial^r}{\partial t^r} \quad (17c, d)$$

$S_{ij}$  and  $e_{ij}$  are the stress and strain deviators, respectively, and where  $\sigma_{kk}$  and  $\epsilon_{kk}$  are three times the mean normal stress and strain. The coefficients  $p_r$ ,  $p_r'$ ,  $q_r$ ,  $q_r'$  are constants. The stress and strain deviators are related to the stress tensor  $\sigma_{ij}$  and the strain tensor  $\epsilon_{ij}$  by the relations

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \quad (18a)$$

$$e_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_{kk} \delta_{ij} \quad (18b)$$

where  $\delta_{ij}$  is the Kronecker delta, equal to unity when  $i = j$  and equal to zero when  $i \neq j$ .

Just as each stress and displacement component is a scalar function, so is each of the stress and strain deviators. Hence, it is permissible to take the Fourier transform of equation (16) and write

$$P_s(j\omega) \mathcal{F}(S_{ij}) = Q_s(j\omega) \mathcal{F}(e_{ij}) \quad (19a)$$

$$P_\nu'(j\omega) \mathcal{F}(\sigma_{ii}) = Q_\nu'(j\omega) \mathcal{F}(\epsilon_{ii}) \quad (19b)$$

In order to rewrite the previous relations in such a form that they appear similar to Hooke's law, let us define the viscoelastic complex moduli  $G_c(j\omega)$ ,  $K_c(j\omega)$  as

$$G_c(j\omega) = \frac{1}{2} Q_s(j\omega) / P_s(j\omega) = \frac{1}{2} \mathcal{F}(S_{ij}) / \mathcal{F}(e_{ij}) \quad (20a)$$

$$K_c(j\omega) = \frac{1}{3} Q_\nu'(j\omega) / P_\nu'(j\omega) = \frac{1}{3} \mathcal{F}(\sigma_{ii}) / \mathcal{F}(\epsilon_{ii}) \quad (20b)$$

Then the transformed viscoelastic stress-strain law may be written

$$\mathcal{F}(S_{ij}) = 2G_c \mathcal{F}(e_{ij}) \quad (21a)$$

$$\mathcal{F}(\sigma_{ii}) = 3K_c \mathcal{F}(\epsilon_{ii}) \quad (21b)$$

The previous relations would look like Hooke's law for an elastic material if stress and strain were to replace the Fourier transform of stress and strain, if the shear modulus  $G$  were to replace the complex shear modulus  $G_c(j\omega)$  and if the bulk modulus  $K$  were to replace the complex bulk modulus  $K_c(j\omega)$ .

The foregoing analogy suggests an approach to the vibration problem in which the core is made of a viscoelastic material. The approach is based on the analogy and the fact that the strain-displacement relations and the equations of motion of an assembly with an elastic core or a viscoelastic core are the same; the only difference arises in the constitutive equations; namely, Hooke's law or the viscoelastic relation of equation (16). The Fourier transform of Hooke's law has been shown to correspond to the Fourier transform of the linear viscoelastic law, equation (16), provided  $G$  is replaced by  $G_c$  and  $K$  by  $K_c$ . Then all equations are identical and likewise their solution. Hence the solution to an elastic vibration problem can be converted to a solution of a viscoelastic vibration problem, by first taking the Fourier transform of stress, strain, and displacement followed by the replacement of the shear modulus  $G$  by the complex shear modulus  $G_c(j\omega)$ , and the replacement of modulus  $K$  by the complex bulk modulus  $K_c(j\omega)$ . The resulting expressions describe the spectra of stress, strain, and displacement associated with the viscoelastic problem.

The Fourier transform of stress, strain, and displacement associated with elastic solutions to the vibration problem under consideration were presented in the previous sections. They can be converted to viscoelastic solutions merely by replacing  $G$  with  $G_c$  and  $K$  by  $K_c$ , where  $G_c$  and  $K_c$  are the viscoelastic complex moduli of the core material.

## Displacement Spectra for Lowest Wave Number

The general results derived in the previous sections will be specialized for the lowest wave number  $n = 0$ , in order to present an explicit solution to the spectra of displacement components.

When  $n = 0$ , equation (7) for  $U_n$ ,  $V_n$  reduces to

$$U_0 = -\alpha c_0 J_1(\alpha r); \quad V_0 = -\beta b_0 J_1(\beta r) \quad (22a, b)$$

where the constants  $b_0$ ,  $c_0$  are the solutions to the two algebraic relations of equation (9) which reduce to

$$M \begin{bmatrix} b_0 \\ c_0 \end{bmatrix} = \frac{a^5}{D} \begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} \quad (23)$$

The matrix  $M$  for  $n = 0$  is

$$M = A \begin{bmatrix} J_0(\zeta) & 0 \\ 0 & J_0(z) \end{bmatrix} - B \begin{bmatrix} \zeta J_1(\zeta) & 0 \\ 0 & z J_1(z) \end{bmatrix} \quad (24)$$

where the matrices  $A$  and  $B$  are

$$A = \begin{bmatrix} 0 & \left(K + \frac{4}{3}G\right) z^2 a^3/D \\ -\zeta^2 G a^3/D & 0 \end{bmatrix} \quad (25a)$$

$$B = \begin{bmatrix} 0 & \lambda\omega^2 + \frac{2Ga^3}{D} - 12\left(\frac{a}{h}\right)^2 \\ -\lambda\omega^2 - 2Ga^3/D & 0 \end{bmatrix} \quad (25b)$$

Hence the constants are prescribed by the relations

$$\beta b_0 = \frac{aQ_0}{(\rho\omega^2 a + 2G)J_1(\zeta) - G\zeta J_0(\zeta)} \quad (26a)$$

$$\alpha c_0 = \frac{aP_0}{\left(K + \frac{4}{3}G\right) zJ_0(z) - \left[\rho a\omega^2 + 2G - \frac{Eh}{a(1-\nu^2)}\right] J_1(z)} \quad (26b)$$

The constant  $b_0$  is seen to depend only on  $Q_0$  whereas the constant  $c_0$  is dependent only upon  $P_0$ . When equation (26) is substituted into equation (22), the spectra of displacement components are obtained.

$$U_0 = \frac{aP_0 J_1(\alpha r)}{\left[\rho a\omega^2 + 2G - \frac{Eh}{a(1-\nu^2)}\right] J_1(z) - \left(K + \frac{4}{3}G\right) zJ_0(z)} \quad (27a)$$

$$V_0 = \frac{aQ_0 J_1(\beta r)}{G\zeta J_0(\zeta) - (\rho a\omega^2 + 2G)J_1(\zeta)} \quad (27b)$$

The displacement spectra are explicit functions of  $\omega$ ,  $r$  for prescribed values of  $P_0$ ,  $Q_0$ .

The foregoing results show that the radial displacement depends only on the radial pressure, while the tangential displacement depends only on the tangential surface shear. One observes, also from equation (27a) that the term  $U_0$  vanishes as the bulk modulus becomes infinitely large. Hence a nonzero radial displacement cannot be independent of  $\theta$  in a solid incompressible core in a state of plane strain. Also, with  $V_0(j\omega, r)$  independent of the bulk modulus  $K$ , it follows that the response of the assembly to a tangential shear which does not vary in the circumferential direction is the same for an incompressible as it is for a compressible core material.

It is interesting to note that the forced-vibration response to a tangential shear  $q(\theta, t)$  which is independent of  $\theta$ , without the presence of any radial pressure  $p(\theta, t)$ , results in a rigid-body rotation of the casing about the assembly axis. The response to a time-varying radial pressure which is uniform around the circumference results in radial "breathing" of the assembly.

The spectra of displacement components  $U_0$ ,  $V_0$  give the response to surface load spectra  $P_0$ ,  $Q_0$  corresponding to a radial pressure uniform around the circumference of the casing, and a uniform shear flow, respectively. The radial pressure has no resultant force or moment, while the shear flow is statically equivalent to a twisting couple.

## Discussion of Results

It is difficult to evaluate the significance of the derived results without considering very specific situations. Hence consideration will be given to the behavior of assemblies with two different viscoelastic cores, one a Voigt material and the other a Maxwell material. The characteristics of each of these visco-

elastic materials will be varied so that the effect of changing retardation time and relaxation time can be evaluated.

First let us examine the behavior of an assembly in which an elastic case is bonded to a viscoelastic solid core which is assumed to be elastic in dilatation, but acts like a Voigt solid in distortion. Our objective is to determine the frequency response of the radial bond stress  $\sigma_{rr}(a, \theta, t)$  due to a lateral pressure loading  $p(\theta, t)$ .

It is to be recalled that the general frequency-response function  $\Sigma_n^{(rr)}(j\omega, a)/P_n(j\omega)$  for the aforementioned viscoelastic problem can be obtained from equations (7), (9), (10), (11), and (15b) by setting  $Q_n = 0$  and replacing  $G$  by the complex shear modulus

$$G_c = G_0(1 + j\omega\tau) \quad (28)$$

where  $\tau$  is the Voigt retardation time and  $G_0$  is the static modulus of rigidity.

At the lowest circumferential wave number,  $n = 0$  the response can be cast in the form

$$\frac{\Sigma_0^{(rr)}(j\omega, a)}{P_0(j\omega)} = \frac{zJ_0(z) - 2(C_2/C_1)^2 J_1(z)}{\left\{ \frac{\rho}{h\gamma} \left(\frac{h}{a}\right) z^2 + \left(\frac{C_2}{C_1}\right)^2 \left[ 2 - \frac{E}{G_c(1-\nu^2)} \left(\frac{h}{a}\right) \right] \right\} J_1(a) - zJ_0(z)} \quad (29)$$

where the dimensionless frequency

$$z = \omega a/C_1 \quad (30)$$

and the terms

$$C_1 = \sqrt{K + \frac{4}{3}G_c}/\gamma; \quad C_2 = \sqrt{G_c}/\gamma \quad (31)$$

One should observe that when  $h/a$  vanishes, the ratio  $\Sigma_0^{(rr)}(j\omega, a)/P_0(j\omega)$  is equal to  $-1$  at all frequencies  $\omega$ . This occurs because of the boundary conditions at  $r = a$ , which force the surface pressure  $p(\theta, t)$  to be equal to  $-\sigma_{rr}(a, \theta, t)$  when the casing is nonexistent.

When equation (28) is introduced into equations (30) and (31) the dimensionless frequency  $z$  is found to be

$$z = \frac{\omega a(C_2^{(0)}/C_1^{(0)})}{\frac{\tau}{a} C_2^{(0)} \sqrt{1 + \frac{4}{3}(C_2^{(0)}/C_1^{(0)})^2 j\omega\tau}} \quad (32)$$

The ratio

$$\left(\frac{C_2}{C_1}\right)^2 = \frac{(C_2^{(0)}/C_1^{(0)})^2}{1 + \left[\frac{4}{3}(C_2^{(0)}/C_1^{(0)})^2 \omega\tau\right]^2} \times \left\{ 1 + \omega\tau \left(\frac{4}{3}\left(\frac{C_2^{(0)}}{C_1^{(0)}}\right)^2 \omega\tau + j \left[ 1 - \frac{4}{3}\left(\frac{C_2^{(0)}}{C_1^{(0)}}\right)^2 \right] \right) \right\} \quad (33)$$

where  $C_1^{(0)}$ ,  $C_2^{(0)}$  are the zero frequency (static) values of  $C_1$ ,  $C_2$ , respectively. Also,

$$\frac{E}{G_c(1-\nu^2)} = \frac{E}{G_0(1-\nu^2)} \left( \frac{1}{1 + j\omega\tau} \right) \quad (34)$$

Inspection of equations (29), (32), (33) and (34) shows that the frequency-response function depends only on six dimensionless real parameters,

$$\omega\tau, \frac{\tau}{a} C_2^{(0)}, \frac{C_2^{(0)}}{C_1^{(0)}}, \frac{E}{G_0(1-\nu^2)}, \frac{h}{a} \quad \text{and} \quad \frac{\rho}{h\gamma}$$

The first parameter is proportional to the excitation frequency  $\omega$ . The second parameter is proportional to the core retardation time  $\tau$ . The third parameter and the fourth are a measure of the elastic properties of the assembly. The fifth parameter is the thickness to radius ratio, and the sixth parameter is the ratio of casing density to propellant density.

Let us continue our examination of a Voigt core by computing for the following fixed values of the independent parameters

$$\frac{h}{a} = 0.1, \quad \frac{E}{G_0(1-\nu^2)} = 22,500,$$

$$\frac{C_1^{(0)}}{C_2^{(0)}} = 30.35, \quad \frac{\rho}{h\gamma} = 1.96,$$

and for two values of the core's nondimensional viscosity,

$$\frac{\tau C_2^{(0)}}{a} = 0, \quad \text{and} \quad \frac{\tau C_2^{(0)}}{a} = 0.2455$$

Consideration is given only to the three lowest natural frequencies of the assembly. The results are obtained from equation (29) and plotted in Fig. 2 and Fig. 3 in the form of  $|\Sigma_0^{(rr)}/P_0|$  versus the dimensionless frequency  $\omega a/C_1^{(0)}$  and plotted in Fig. 4 and Fig. 5 as the phase angle  $\angle \Sigma_0^{(rr)}/P_0$  versus the dimensionless frequency  $\omega a/C_1^{(0)}$  for each value of  $\tau C_2^{(0)}/a$ .

At first the Bessel functions of complex argument  $z$  in equation (29) were obtained using the tabulated values of Reference [9]. However these tables list  $J_0(z)$  and  $J_1(z)$  for increments in  $\angle z$  of 5 deg, which is too coarse a mesh size for lightly damped incompressible materials. A more convenient method of calculation was obtained from equation (32) based upon the small magnitude of the phase angle  $\angle z$ ; namely,

$$\angle z = -\frac{1}{2} \tan^{-1} \left[ \frac{4}{3} \left( \frac{C_2^{(0)}}{C_1^{(0)}} \right)^2 \omega \tau \right] \quad (35)$$

Derivation of the simplified form is aided by the observations that for the numerical values and frequency range used in the example  $0 > \angle z > -0.05$  rad. Rather than use tabulated values, one expands  $J_1(z)$  and  $zJ_0(z)$  in power series in the real variable  $\angle z$ , holding  $|z|$  fixed. These series converge in some interval about  $\angle z = 0$ . When only the linear terms in  $\angle z$  are retained the power series may be written

$$zJ_0(z) = xJ_0(x) + jx[J_0(x) - xJ_1(x)] \angle z \quad (36a)$$

$$J_1(z) = J_1(x) + j[xJ_0(x) - J_1(x)] \angle z \quad (36b)$$

with  $|z| = x = \omega a/C_1^{(0)}$

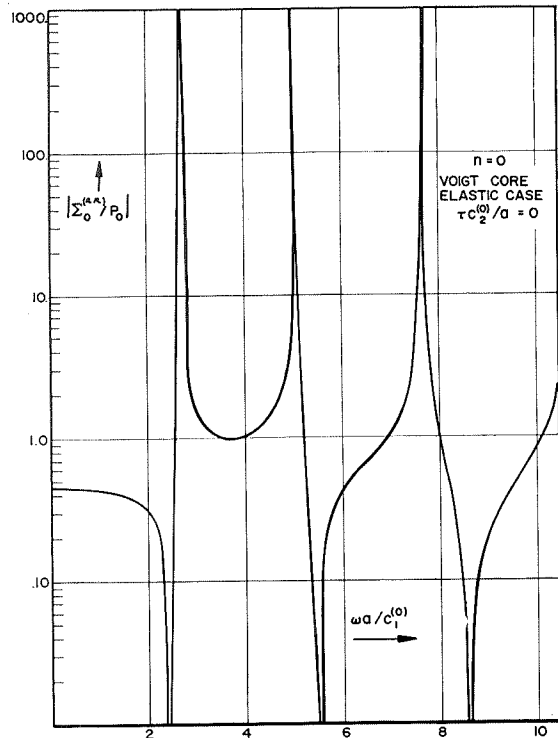


Fig. 2 Amplitude versus frequency response of radial bond stress due to lateral pressure

It is important to observe that the approximation we make assumes  $[C_2^{(0)}/C_1^{(0)}] \omega \tau \ll 1$ , which is correct for the particular numerical value of  $C_1^{(0)}/C_2^{(0)} = 30.35$  used in the example. However  $\omega \tau$  is definitely not less than unity.

Fig. 2, which shows the stress amplitude ratio as a function of the dimensionless frequency for vanishing core viscosity, clearly illustrates the infinite resonant spikes at  $\omega a/C_1^{(0)} = 2.729, 5.035, 7.739$  which are the natural frequencies of the all-elastic assembly, previously discussed by the authors [1]. It is interesting to observe the antiresonances at  $\omega a/C_1^{(0)} = 2.40, 5.55, 8.63$ . Theoretically a surface pressure of arbitrary amplitude, uniformly distributed around the circumference, but sinusoidal in time at these antiresonant frequencies would not induce any radial stress at the interface.

The low-frequency asymptote  $|\Sigma_0^{(rr)}/P_0| = 0.450$  represents the static radial stress at the interface due to a uniform surface pressure which is constant in time. A general expression for the static bond stress can be obtained from equation (29) by taking the limit of it as  $\omega$  vanishes; namely,

$$-\frac{\Sigma_0^{(rr)}(0, a)}{P_0(0)} = \frac{1 - \left( \frac{C_2^{(0)}}{C_1^{(0)}} \right)^2}{1 - \left( \frac{C_2^{(0)}}{C_1^{(0)}} \right)^2 + \frac{E}{G_0(1-\nu^2)} \left( \frac{C_2^{(0)}}{C_1^{(0)}} \right)^2 \frac{h}{2a}} \quad (37)$$

It should be observed that the static radial bond stress is always less in magnitude than the surface pressure, and has the same sign as the surface pressure. This response is due to the ability of the case to take hoop stress.

Fig. 3 illustrates the major effects of core viscosity. It is based on the same parameters as Fig. 2, except that  $\tau C_2^{(0)}/a$  is increased from 0 to 0.2455. The infinite resonance spikes are reduced to finite maxima, and the antiresonances are increased in magnitude from zero to local minima. The calculated resonance peaks are  $|\Sigma_0^{(rr)}/P_0| = 12.3, 3.04, 1.43$ , and the antiresonances are  $|\Sigma_0^{(rr)}/P_0| = 0.028, 0.240, 0.260$  in order of increasing frequency.

The effect of core viscosity on attenuating the resonance peaks,

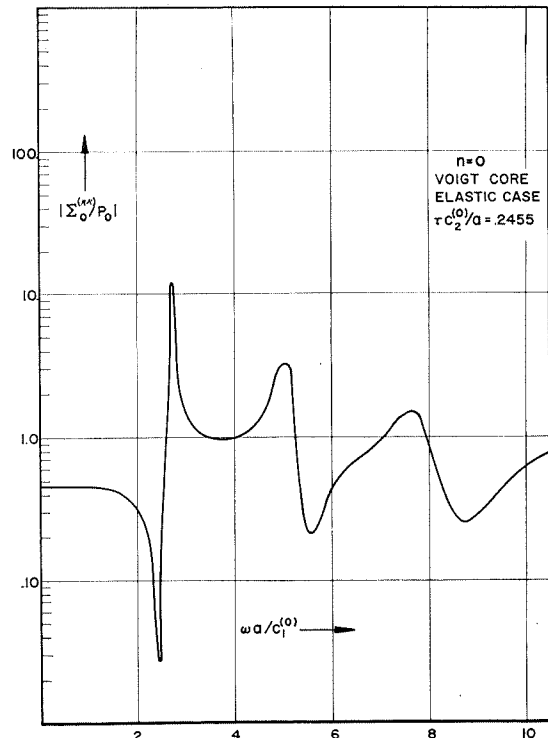


Fig. 3 Amplitude versus frequency response of radial bond stress due to lateral pressure

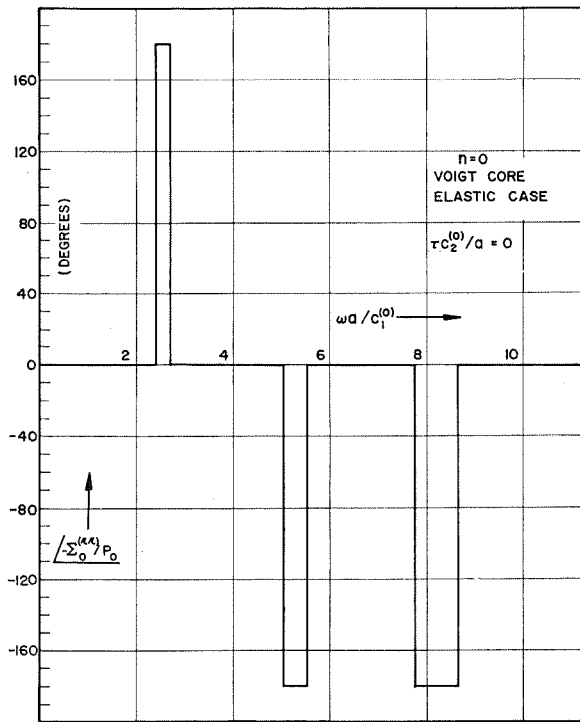


Fig. 4 Phase versus frequency response of radial bond stress due to lateral pressure

and on magnifying the antiresonances increases with increasing frequency. This is to be expected with a Voigt core material, since a Voigt material behaves like an elastic solid for low frequencies, and like a viscous fluid for high frequencies. If the core material were to behave like a Maxwell body in distortion, that is

$$G_c = \frac{j\omega\tau G_0}{1 + j\omega\tau} \quad (38)$$

more complicated effects would occur.

The phase response for an all-elastic assembly may be seen in Fig. 4. Since the frequency response for an all-elastic assembly is always real, the phase is always real, the phase is always plus 180 or minus 180 deg, with jumps in the phase angle  $\angle \Sigma_0^{(rr)}/P_0$  when passing through a resonance or antiresonance.

It can be seen in Fig. 5 that increasing the dimensionless re-

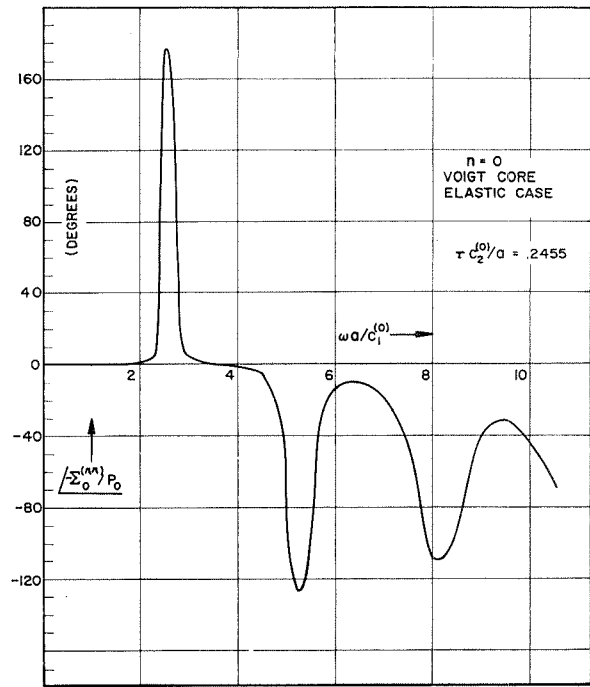


Fig. 5 Phase versus frequency response of radial bond stress due to lateral pressure

tardation time to  $\tau C_2^{(0)}/a = 0.2455$ , rounds the edges off the discontinuous phase response of an all-elastic assembly. It is interesting to note from a comparison of Fig. 2 and Fig. 4, as well as Fig. 3 and Fig. 5 that extremals in phase occur midway between extremals in gain.

In order to obtain a comparison of Voigt and Maxwell core materials, the first three resonance peaks of  $|\Sigma_0^{(rr)}/P_0|$  are evaluated for increasing values of  $\tau C_2^{(0)}/a$ . Results are listed in Table 1.

It may be observed from this table that for both Voigt and Maxwell cores, and small values of  $\tau$  the amplitude of  $\Sigma_0^{(rr)}/P_0$  decreases with increasing  $\tau$  at all three resonant frequencies. As  $\tau$  gets larger, the resonant amplitude increases for the Maxwell material, while the Voigt material continues to exhibit decreasing resonant amplitudes.

This phenomenon can be explained on the basis of amplitude

Table 1 Comparison of resonant amplitudes of  $\Sigma_0^{(rr)}/P_0$  for Voigt and Maxwell core materials

$$\rho/h/\gamma = 1.96 \quad E/G_0(1 - \nu^2) = 22,500 \quad C_1^{(0)}/C_2^{(0)} = 30.35 \quad h/a = 0.1$$

$$\left. \begin{array}{l} \text{Voigt: } G_c = G_0(1 + j\omega\tau) \\ \text{Maxwell: } G_c = G_0 j\omega\tau / (1 + j\omega\tau) \end{array} \right\} \quad \begin{array}{l} C_1^{(0)} = \sqrt{K + \frac{4}{3} G_0/\gamma} \\ C_2^{(0)} = \sqrt{G_0/\gamma} \end{array}$$

				First resonant frequency		$ \Sigma_0^{(rr)}/P_0 $		
$\tau C_2^{(0)}/a$	$\omega a/C_1^{(0)}$					0.001	0.1	0.2455
Maxwell	2.729	2.729	2.729	2.729	302,353	3044	2108	5114
Voigt	2.729	2.729	2.729	2.729	302,845	3028	30.28	12.3
				Second resonant frequency		$ \Sigma_0^{(rr)}/P_0 $		
$\tau C_2^{(0)}/a$	$\omega a/C_1^{(0)}$					0.001	0.1	0.2455
Maxwell	5.032	5.032	5.034	5.034	79,373	812	1863	4558
Voigt	5.034	5.034	5.039	5.065	79,469	795	7.87	3.04
				Third resonant frequency		$ \Sigma_0^{(rr)}/P_0 $		
$\tau C_2^{(0)}/a$	$\omega a/C_1^{(0)}$					0.001	0.1	0.2455
Maxwell	7.733	7.734	7.739	7.739	38,136	402	2108	5168
Voigt	7.739	7.739	7.755	7.849	38,153	382	3.77	1.43

$|G_c|$  (modulus) and phase angle  $\angle G_c$  (argument) of the complex shear modulus of rigidity  $G_c$ .

Voigt:

$$|G_c| = G_0 \sqrt{1 + (\omega\tau)^2} \quad (39a)$$

$$\angle G_c = \tan^{-1} \omega\tau \quad (39b)$$

Maxwell:

$$|G_c| = G_0 / \sqrt{1 + (1/\omega\tau)^2} \quad (39c)$$

$$\angle G_c = \tan^{-1} (1/\omega\tau) \quad (39d)$$

Since viscosity enters into the cylinder assembly only through  $G_c$ , the amount of damping introduced will be proportional to both  $|G_c|$  and  $\angle G_c$ .

For the Voigt material, amplitude and phase angle of  $G(j\omega)$  each increase with increasing  $\tau$  for fixed resonant frequency  $\omega$ , so that for sufficiently large  $\tau$  the core acts like a viscous fluid. Moreover at  $\tau = 0$  the Voigt material has  $G_c(j\omega) = G_0$ ; i.e., it behaves like an elastic solid. Hence, at each resonant frequency, damping increases monotonically with  $\tau$  for a Voigt material.

For the Maxwell material, amplitude  $|G_c|$  increases with increasing  $\tau$ , for fixed resonant frequency  $\omega$ ; while  $\angle G_c$  decreases. In the limit, as  $\tau \rightarrow \infty$ , the Maxwell material has  $G_c(j\omega) = G_0$ ; i.e., it behaves like an elastic solid. At  $\tau = 0$ ,  $G_c(j\omega) = 0$ , so that for sufficiently small  $\tau$ , the Maxwell material acts like an inviscid fluid. Hence, for each resonant frequency, there must be an optimum  $\tau$ , at which the core material exhibits the most damping. Moreover, since  $G_c$  is a function of  $\omega$  only through the product  $\omega\tau$ , the optimum  $\tau$  must decrease as the resonant frequency  $\omega$  increases. This theory is borne out by the results shown in Table 1.

## Summary and Conclusion

In this paper we have reviewed the equations governing the forced, transverse vibration of a cylindrical assembly consisting of a solid core with a case-bonded outer shell.

Using a Fourier transform method of solution, the frequency response of the assembly was obtained, considering both an arbitrary normal pressure and tangential shear applied to the outer surface of the casing. The solution was shown to hold for a general linear viscoelastic core, if the elastic constants are replaced by corresponding complex moduli.

Curves of the lowest circumferential wave-number frequency response  $\Sigma_0^{(rr)}(j\omega, a)/P_0(j\omega)$  of the radial bond stress due to lateral pressure were plotted for a particular configuration. Several values of core time constant were used, assuming a Voigt model and also a Maxwell model.

The results of the present analysis were presented in terms of the Fourier transform of the stress and displacement components. One can evaluate each of these transforms for specific time-dependent loading histories. Then, by the use of the inverse transform, it is possible to determine the time-dependent relations for stress and displacement.

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